

ON EMBEDDINGS WITH LOCALLY NICE CROSS-SECTIONS⁽¹⁾

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Abstract. A k -dimensional compactum X^k in euclidean space E^n ($n-k \geq 3$) is said to be locally nice in E^n if $E^n - X^k$ is 1-ULC. In this paper we prove a general theorem which implies, in particular, that X^k is locally nice in E^n if the intersection of X^k with each horizontal hyperplane of E^n is locally nice in the hyperplane. From known results we obtain immediately that a k -dimensional polyhedron P in E^n ($n-k \geq 3$ and $n \geq 5$) is tame in E^n if each $(E^{n-1} \times \{w\}) - P$ ($w \in E^1$) is 1-ULC. However, by strengthening our general theorem in the case $n=4$, we are able to prove this result for $n=4$ as well. For example, an arc A in E^4 is tame if each horizontal cross-section of A is tame in the cross-sectional hyperplane (that is, lies in an arc that is tame in the hyperplane).

In this note we give a sufficient condition that a k -dimensional compactum X in euclidean n -space E^n ($n-k \geq 3$) have a 1-ULC complement in E^n . A principal application of our result is that $E^n - X$ is 1-ULC provided each $(E^{n-1} \times \{w\}) - X$ ($w \in E^1$) is 1-ULC, where we consider E^n as $E^{n-1} \times E^1$.

THEOREM 1. Suppose that X is a k -dimensional compactum in $E^{n+1} = E^n \times E^1$ ($n \geq 3$ and $n-k \geq 2$) such that for each $w \in E^1$ and each $\varepsilon > 0$ there exists an ε -push h of (E^{n+1}, X) (see [4]) such that $(E^n \times \{w\}) - h(X)$ is 1-ULC. Then E^{n+1} is 1-ULC.

THEOREM 2. Suppose that X is a 1-dimensional compactum in E^4 satisfying the hypothesis of Theorem 1 and having the additional property that, for some $\delta > 0$, no component of $(E^3 \times \{w\}) \cap X$ contains a nontrivial (Čech) 1-cycle of diameter less than δ for any $w \in E^1$.

Then for each 2-complex K in E^4 and for each $\varepsilon > 0$, there exists an ε -push g of $(E^4, X \cap K)$ such that $g(K) \cap X = \emptyset$.

COROLLARY 1. Suppose that $h_t: M \rightarrow Q$ ($t \in [0, 1]$) is an isotopy of locally flat embeddings of the topological m -manifold M in the interior of the topological q -manifold Q ($q-m \geq 3$). Then the embedding $H: M \times I \rightarrow Q \times I$ defined by $H(x, t) = (h_t(x), t)$ is a locally flat embedding.

Received by the editors May 7, 1970.

AMS 1969 subject classifications. Primary 5478, 5570.

Key words and phrases. Locally nice embeddings, 1-ULC subsets of E^n , tame embeddings, embeddings with tame cross-sections, embeddings with locally nice cross-sections, topological embeddings of compacta, topological embeddings of polyhedra.

(¹) Partially supported by NSF grant GP-19964.

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COROLLARY 2. *Suppose that A is an arc in E^{n+1} ($n \geq 3$) such that $A \cap (E^n \times \{w\})$ has a 1-ULC complement in $E^n \times \{w\}$. Then A is tame.*

More generally, if for each $w \in E^1$ there exists a small push h of (E^{n+1}, A) such that $(E^n \times \{w\}) - h(A)$ is a tame subset of $E^n \times \{w\}$ of dimension ≤ 0 , then A is tame.

COROLLARY 3. *If P is a k -dimensional polyhedron in E^{n+1} ($n \geq 3$ and $n - k \geq 2$) such that $(E^n \times \{w\}) - P$ is 1-ULC for each $w \in E^1$, then P is tame.*

COROLLARY 4. *Suppose that X is a k -dimensional compactum in E^{n+1} ($n \geq 3$, $2k + 1 \leq n$) and that $f: X \rightarrow E^{n+1}$ is an embedding such that both X and $f(X)$ satisfy the hypotheses of Theorem 1 if $n \geq 4$ and Theorem 2 if $n = 3$. Then there exist a compact set C and an isotopy h_t ($t \in [0, 1]$) of E^{n+1} such that $h_0 = \text{identity}$, $h_1|_X = f$, and each h_t is the identity outside C .*

The proofs of the corollaries in the case $n \geq 4$ are each obtained as a direct application of Theorem 1 and the results of [2] and [3]. The cases in which $n = 3$ follow from Theorem 2 and the methods of [1].

Let D^2 denote the unit disk in E^2 and let $S^1 = \text{Bd } D^2$. A subset U of E^n is said to be 1-ULC if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that every map $f: S^1 \rightarrow U$ with $\text{diam } f(S^1) < \delta$ extends to a map $F: D^2 \rightarrow U$ with $\text{diam } F(D^2) < \varepsilon$. We use I to denote the interval $[-1, 1]$ considered as a subset of the first factor of E^n . Let

$$D^* = \{(x, t) \in E^n \times E^1 \mid x \in I \text{ and } 0 \leq t \leq 1\}.$$

LEMMA 1. *Suppose that $E^n - X^k$ ($n - k \geq 3$) is 1-ULC and that $F: D^2 \rightarrow E^n$ is a map with $F(S^1) \cap X = \emptyset$. Then for each $\varepsilon > 0$ there exists a map $G: D^2 \rightarrow E^n - X$ such that $d(F(x), G(x)) < \varepsilon$ and $G|_{S^1} = F|_{S^1}$.*

For a proof of this lemma, see the proof of Lemma 1 of [1].

LEMMA 2. *Suppose that $X^k \subset E^{n+1} = E^n \times E^1$ ($n \geq 3$ and $n - k \geq 2$) and that $f: D^* \rightarrow E^n \times [0, 1]$ is a level-preserving map (that is, $f(x, t) \in E^n \times \{t\}$ for each $(x, t) \in D^*$) such that $f|_{\text{Bd } I \times [0, 1]} = \text{inclusion}$ and $f(\text{Bd } D^*) \cap X = \emptyset$. Then for each neighborhood U of $f(D^*)$ in E^{n+1} , there exist numbers*

$$t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$$

and embeddings $\alpha_i: I \rightarrow E^n$ ($i = 1, 2, \dots, m$) such that

- (a) $\alpha_i|_{\text{Bd } I} = \text{inclusion}$,
- (b) $(\alpha_1(x), 0) = f(x, 0)$ and $(\alpha_m(x), 1) = f(x, 1)$, and
- (c) $\alpha_i(I) \times [t_{i-1}, t_i] \subset U - X$ for each $i = 1, 2, \dots, m$.

Proof. Let f_t be $f|_{I \times \{t\}}$ followed by the projection into E^n . We may assume that f_0 and f_1 are embeddings ($n \geq 3$). Since $f(I \times \{0\}) \cap X = f(I \times \{1\}) \cap X = \emptyset$, there exist numbers $0 < s_0 \leq s_1 < 1$ such that $(f_0(I) \times [0, s_0]), (f_1(I) \times [s_1, 1]) \subset U - X$. For each $t \in [s_0, s_1]$ there exists an embedding $\alpha_t: I \rightarrow E^n$ such that $\alpha_t|_{\text{Bd } I} = \text{inclusion}$ and $\alpha_t(I) \times \{t\} \subset U - X$ (since $n - k \geq 2$). Hence, there exists $\delta_t > 0$ such that $\alpha_t(I) \times (t - \delta_t, t + \delta_t) \subset U - X$.

Let $\lambda > 0$ be a Lebesgue number for the cover $\{(t - \delta_i, t + \delta_i) \mid s_0 \leq t \leq s_1\}$ of $[s_0, s_1]$. Choose $t_1 = s_0 < t_2 < \dots < t_{m-1} = s_1$ so that $t_i - t_{i-1} < \lambda$ ($i = 2, \dots, m-1$). Taking $\alpha_1 = f_0$ and $\alpha_m = f_1$, we see that for $i = 1, \dots, m$ there exist embeddings $\alpha_i: I \rightarrow E^n$ satisfying (a), (b), and (c).

LEMMA 3. Suppose that $X^k \subset E^{n+1} = E^n \times E^1$ ($n \geq 3$ and $n - k \geq 2$) satisfies the hypothesis of Theorem 1 and that $f: D^* \rightarrow E^n \times [0, 1]$ is a level-preserving map with $f|_{\text{Bd } I \times [0, 1]} = \text{inclusion}$ and $f(\text{Bd } D^*) \cap X = \emptyset$. Then for each neighborhood U of $f(D^*)$ there exists a map $g: D^* \rightarrow U - X$ such that $g|_{\text{Bd } D^*} = f|_{\text{Bd } D^*}$.

Proof. Let U be a neighborhood of D^* and assume that $f|_{I \times \{0, 1\}}$ is an embedding. From Lemma 2 we obtain numbers $t_0 = 0 < t_1 < t_2 < \dots < t_m = 1$ and embeddings $\alpha_i: I \rightarrow E^n$ ($i = 1, 2, \dots, m$) satisfying

$$\begin{aligned}\alpha_i|_{\text{Bd } I} &= \text{inclusion}, \\ \alpha_1 &= f_0 \quad \text{and} \quad \alpha_m = f_1\end{aligned}$$

(again, f_i is $f|_{I \times \{t_i\}}$ followed by the projection into E^n), and

$$\alpha_i(I) \times [t_{i-1}, t_i] \subset U - X$$

for each $i = 1, 2, \dots, m$. Moreover, we may assume that each

$$[\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$$

is a simple closed curve that bounds a singular disk D'_i in $U \cap (E^n \times \{t_i\})$ ($i = 1, \dots, m-1$).

By hypothesis, there exists a small push h_1 of (E^{n+1}, X) such that

$$(E^n \times \{t_1\}) - h_1(X)$$

is 1-ULC. If h_1 is a sufficiently small push, then we will have

$$h_1|_{\bigcup_{i=1}^m (\alpha_i(I) \times [t_{i-1}, t_i])} = \text{identity}.$$

By Lemma 1, we may replace D'_1 by a singular disk D''_1 in $[U \cap (E^n \times \{t_1\})] - h_1(X)$. Then $h_1^{-1}(D''_1) \cap X = \emptyset$ and $\text{Bd } (h_1^{-1}(D''_1)) = (\alpha_1(I) \cup \alpha_2(I)) \times \{t_1\}$. Thus

$$[\alpha_1(I) \cup \alpha_2(I)] \times \{t_1\}$$

bounds a disk $D_1 (= h_1^{-1}(D''_1))$ in $U - X$.

Continuing in this manner, we obtain singular disks $D_1, D_2, \dots, D_{m-1}$ in $U - X$ such that $\text{Bd } (D_i) = [\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$ for $i = 1, \dots, m-1$. Taking the union

$$(\alpha_1(I) \times [t_0, t_1]) \cup D_1 \cup (\alpha_2(I) \times [t_1, t_2]) \cup \dots \cup D_{m-1} \cup (\alpha_m(I) \times [t_{m-1}, t_m]),$$

we obtain a singular disk D in $U - X$ such that $\text{Bd } D = f(\text{Bd } D^*)$.

Proof of Theorem 1. Suppose that $X \subset E^{n+1} = E^n \times E^1$ satisfies the hypothesis of Theorem 1. Let Γ' be a small simple closed curve in $E^{n+1} - X$. We may change

Γ' by a small homotopy to a simple closed curve Γ in $E^{n+1} - X$ such that Γ is the union of a finite collection of small line segments, each being parallel to either $E^n \times \{0\}$ or to $\{0\} \times E^1$.

Let v_0 be a vertex of Γ . We are going to construct a small homotopy of Γ in $E^{n+1} - X$ that takes Γ into $E^n \times \{v_0\}$.

Let J be a "horizontal" segment in Γ (that is, J is parallel to $E^n \times \{0\}$), and let u and v be the endpoints of J . Let $f_t: J \rightarrow E^{n+1}$ ($t \in [0, 1]$) be the natural homotopy that slides J into the plane $E^n \times \{v_0\}$, oriented so that $f_0 = \text{inclusion}$ and $f_1(J) \subset E^n \times \{v_0\}$.

Since $n+1 \geq 4$ and $n-k \geq 2$, we may assume that the vertical segments $f(\{u\} \times [0, 1])$ and $f(\{v\} \times [0, 1])$ (where $f(x, t) = f_t(x)$) miss X . (A small argument is required to see that this can be accomplished with a homeomorphism that preserves $(n+1)$ th coordinates.) Since $n-k \geq 2$, we may also assume that $f_1(J) \cap X = \emptyset$ by slightly altering the homotopy f_t on the interior of J as t approaches 1.

Lemma 3 now allows us to replace f_t ($t \in [0, 1]$) by a homotopy $g_t: J \rightarrow E^n - X$ ($t \in [0, 1]$) having the properties that $g_0 = f_0$, $g_1 = f_1$, $g|_{\text{Bd } J \times [0, 1]} = f|_{\text{Bd } J \times [0, 1]}$, and $g(J \times [0, 1])$ lies in any preassigned neighborhood U of $f(J \times [0, 1])$.

Thus the natural homotopy of Γ into the plane $E^n \times \{v_0\}$ can be replaced by a homotopy $h_t: \Gamma \rightarrow E^{n+1} - X$ ($t \in [0, 1]$) that takes Γ into $E^n \times \{v_0\}$. Applying the hypothesis of Theorem 1 once again to the plane $E^n \times \{v_0\}$ as in the proof of Lemma 3, we see that $h_1(\Gamma)$ can be contracted to a point in $E^{n+1} - X$ by a small homotopy. The combination of these two homotopies gives a small homotopy of Γ to a point in $E^{n+1} - X$.

Before we prove Theorem 2, we must prove a strengthened version of Lemma 3. We also need a little more notation. Let

$$B^4 = \{(x_1, x_2, x_3, x_4) \in E^4 \mid -1 \leq x_1, x_2, x_3 \leq 1 \text{ and } 0 \leq x_4 \leq 1\}$$

(notice that D^* is embedded properly in B^4), and let

$$B_t^3 = \{x \in B^4 \mid x_4 = t\} \quad (t \in [0, 1]).$$

LEMMA 4. *Suppose that X is a 1-dimensional compactum in E^4 satisfying the hypothesis of Theorem 1 such that $X \cap \text{Bd } D^* = \emptyset$ and every component of $X \cap B_t^3$ is acyclic for each $t \in [0, 1]$. Then there exists a homeomorphism $h: B^4 \rightarrow B^4$ such that $h|_{\text{Bd } B^4} = \text{identity}$ and $h(D^*) \cap X = \emptyset$.*

Proof. Let $B^3 = \{(x_1, x_2, x_3) \in E^3 \mid -1 \leq x_i \leq 1 \text{ (} i=1, 2, 3)\}$. Let

$$C_1 = \{(x_1, x_2, x_3) \in \text{Bd } B^3 \mid x_1 = 0\},$$

and for each $t \in (0, 1]$, let $C_t = \{tx \mid x \in C_1\}$ and $S_t = \{(-1, 0, 0), (1, 0, 0)\} * C_t$ (where $A * B$ denotes the join of A and B). Each S_t is a 2-sphere in B^3 obtained by suspending the simple closed curve C_t from the points $(-1, 0, 0)$ and $(1, 0, 0)$.

For each $t \in (0, 1)$ the set $X \cap (S_t \times \{t\})$ does not separate $S_t \times \{t\}$. Thus we can find an embedding $\alpha_t: I \rightarrow S_t$ such that $\alpha_t|_{\text{Bd } I} = \text{inclusion}$ and $\alpha_t(I) \times \{t\} \cap X = \emptyset$.

Applying the methods in the proof of Lemma 3, we can find numbers $t_0=0 < t_1 < \cdots < t_m=1$ and embeddings $\alpha_i: I \rightarrow B^3$ ($i=1, \dots, m$) such that

$$\begin{aligned}\alpha_i|_{\text{Bd } I} &= \text{inclusion,} \\ \alpha_1 &= \alpha_m = \text{inclusion,} \\ \alpha_i(I) \times [t_{i-1}, t_i] \cap X &= \emptyset, \\ \alpha_i(I) \cap \alpha_j(I) &= \text{Bd } I \quad \text{if } i \neq j,\end{aligned}$$

and for $i=2, \dots, m-1$, $\alpha_i(I) \subset S_t$ for some $t \in (0, 1)$.

Now fix i ($2 \leq i \leq m-1$) and consider the simple closed curve

$$\Gamma_i = [\alpha_i(I) \cup \alpha_{i+1}(I)] \times \{t_i\}$$

in $B_{t_i}^3$. We may assume that Γ_i is a polygonal curve. Our construction of α_i and α_{i+1} guarantees that Γ_i bounds a nonsingular polyhedral disk D'_i in $B_{t_i}^3$ such that $D'_i \cap \text{Bd } B_{t_i}^3 = \text{Bd } I \times \{t_i\}$. Let h_i be a small push of (E^4, X) such that $(E^3 \times \{t_i\}) - h_i(X)$ is 1-ULC. Let B_i be a 3-cell in $B_{t_i}^3$ containing D'_i properly. From Lemma 1, we see that Γ_i bounds a singular disk in $B_i - h_i(X)$, and so, by Dehn's lemma [5], Γ_i bounds a nonsingular polyhedral disk D_i in $B_i - h_i(X)$.

Since h_i (and, hence, h_i^{-1}) is a stable homeomorphism of E^4 , we may assume that h_i^{-1} is the identity on $(\alpha_i(I) \times [t_{i-1}, t_i]) \cup (\alpha_{i+1}(I) \times [t_i, t_{i+1}])$ and outside $B^4 \cap E^3 \times [\frac{1}{2}(t_{i-1} + t_i), \frac{1}{2}(t_i + t_{i+1})]$. (This is the first time we have really needed stability.)

Observe that the 2-cell

$$D = \alpha_1(I) \times [t_0, t_1] \cup D_1 \cup \alpha_2(I) \times [t_1, t_2] \cup \cdots \cup D_{m-1} \cup \alpha_m(I) \times [t_{m-1}, t_m]$$

is properly embedded in B^4 and that the cell pair (B^4, D) is unknotted. Let $h': B^4 \rightarrow B^4$ be a homeomorphism that is fixed on $\text{Bd } B^4$ and takes D^* to D , and let $h'': B^4 \rightarrow B^4$ be defined by $h''(x) = h_i^{-1}(x)$ if $x \in B^4 \cap E^3 \times [\frac{1}{2}(t_{i-1} + t_i), \frac{1}{2}(t_i + t_{i+1})]$ and $h''(x) = x$ otherwise. Then the homeomorphism $h = h''h': B^4 \rightarrow B^4$ satisfies all of our requirements.

Proof of Theorem 2. Suppose that $X \subset E^4$ satisfies the hypotheses of Theorem 2, $\varepsilon > 0$, and K is a 2-complex in E^4 . Assume that ε is small enough so that $(E^3 \times \{w\}) \cap X$ does not contain any nontrivial (Čech) 1-cycles of diameter less than ε for any $w \in E^1$.

Subdivide the complex K so that each of its simplexes has diameter less than ε .

First move the vertices of K so that no two of them lie in a single horizontal hyperplane. Next move X off the 1-skeleton of K by an isotopy of E^4 that does not change the x_4 -coordinate of any point of E^4 .

Let σ be a 2-simplex of K , let u be the vertex of σ with the smallest x_4 -coordinate and let v be the one with the largest. Let r and s denote the x_4 -coordinates of u and v , respectively. Choose numbers $t_0 > r$ and $t_1 < s$ such that $\sigma \cap X \subset E^3 \times [t_0, t_1]$. Let B be a 4-cell in $E^3 \times [t_0, t_1]$ of diameter less than ε such that the pair

$(B, \sigma \cap (E^3 \times [t_0, t_1]))$ is homeomorphic to (B^4, D^*) (as defined above) by a homeomorphism that takes each horizontal cross-section of B to a horizontal cross-section of B^4 .

We may now apply Lemma 4 to get a homeomorphism $k: B \rightarrow B$ such that $k|_{\text{Bd } B} = \text{identity}$ and $k(\sigma) \cap X = \emptyset$. Moreover, k is an ε -push of $(E^4, \sigma \cap X)$, since k is isotopic to the identity (keeping $\text{Bd } B$ fixed) via the Alexander isotopy.

If we are careful to construct the 4-cells B corresponding to each 2-simplex σ of K so that any two intersect in a subset of the boundary of each, then the ε -pushes k that we obtain will piece together to give the desired ε -push h of $(E^4, K \cap X)$.

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